Lecture 3: Floating Point

Computer Architecture and
Systems Programming
(252-0061-00)

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Last Time: Integers

- Representation: unsigned and signed
- Conversion, casting
  - Bit representation maintained but reinterpreted
- Expanding, truncating
  - Truncating = mod
- Addition, negation, multiplication, shifting
  - Operations are mod 2^w

- "Ring" properties hold
  - Associative, commutative, distributive, additive 0 and inverse
- Ordering properties do not hold
  - $u > 0$ does not mean $u + v > v$
  - $u, v > 0$ does not mean $u \cdot v > 0$

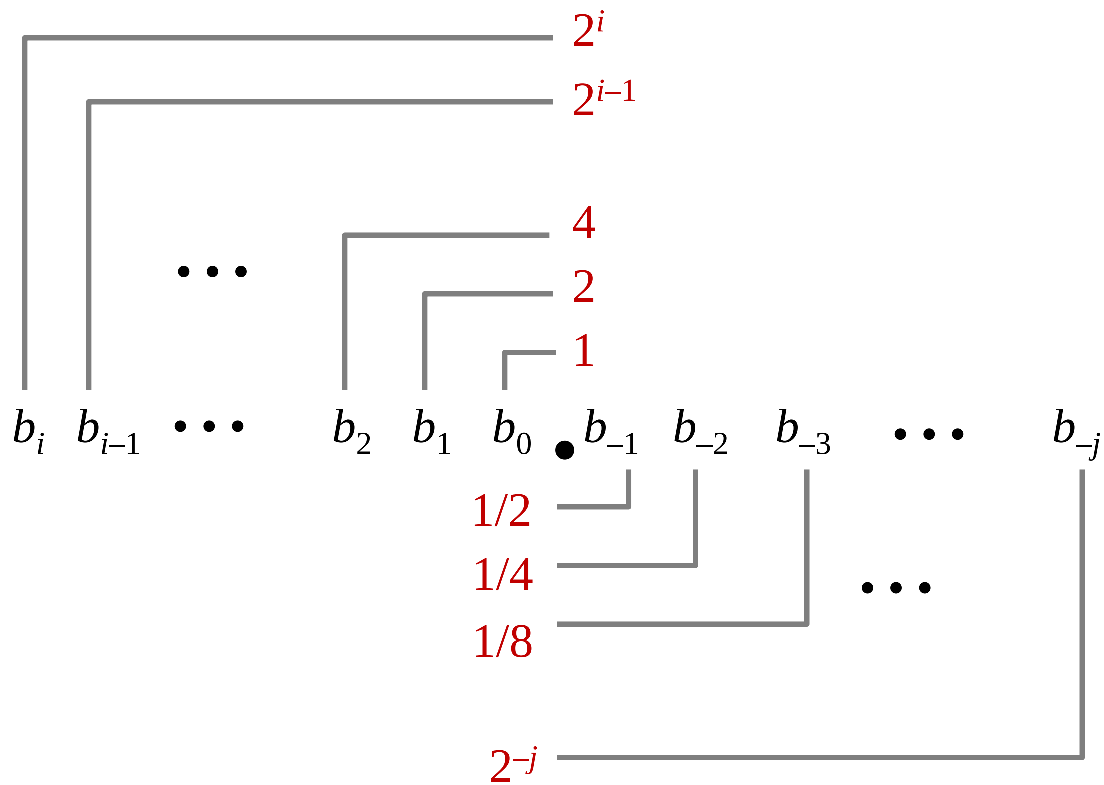
Today: Floating Point

- Background (recap from Digital Design)
  - Fractional binary numbers
  - Definition of IEEE floating point
- More on IEEE floating point
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101?

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number:

$$\sum_{k=-j}^{i} b_k \cdot 2^k$$

Fractional Binary Numbers

Value	Representation
$5^{3}/_{4}$	101.11_2
$2^{7}/_{8}$	10.111_2
$^{63}/_{64}$	0.111111_2

- Observations
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

Value	Representation
1/3	$0.0101010101[01]\ldots_2$
1/5	$0.001100110011[0011]\ldots_2$
1/10	$0.0001100110011[0011]\ldots_2$

You can't represent 0.1_{10}

IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation
(recap from Digital Design)

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- Numerical Form:

$$(-1)^s \, M \, 2^E$$

  - **Sign bit** *s* determines whether number is negative or positive
  - **Significand** *M* normally a fractional value in range [1.0,2.0).
  - **Exponent** *E* weights value by power of two

- Encoding
  - MSB (Most Significant Bit) **`s`** is sign bit *s*
  - **`exp`** field encodes *E* (but is not equal to *E*)
  - **`frac`** field encodes *M* (but is not equal to *M*)

s	exp	frac

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Precisions

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- Single precision: 32 bits

s	exp	frac
1	8	23

- Double precision: 64 bits

s	exp	frac
1	11	52

- Extended precision: 80 bits (Intel only)

s	exp	frac
1	15	63 or 64

Normalized Values

- Condition: **exp** $\neq$ 000…0 and **exp** $\neq$ 111…1

- Exponent coded as **biased** value: ***E = Exp – Bias***
  - *Exp*: unsigned value **exp**
  - *Bias* = 2^{e-1} - 1, where e is number of exponent bits
    - Single precision: 127 (*Exp*: 1…254, *E*: -126…127)
    - Double precision: 1023 (*Exp*: 1…2046, *E*: -1022…1023)

- Significand coded with **implied leading 1:** ***M* = `1.xxx…x`$_2$**
  - **`xxx…x`**: bits of **`frac`**
  - Minimum when **`000…0`** (*M* = 1.0)
  - Maximum when **`111…1`** (*M* = 2.0 – ε)
  - Get extra leading bit for “free”

Normalized Encoding Example

- Value: **`float F = 15213.0;`**

  $15213_{10} = 11101101101101_2$

  $= 1.1101101101101_2 \times 2^{13}$

- Significand
  - $M =$ **1.$\underline{1101101101101}_2$**
  - **`frac`** = **$\underline{1101101101101}0000000000_2$**

- Exponent
  - $E =$ 13
  - $Bias =$ 127
  - $Exp =$ 140 = **10001100**$_2$

- Result:

s	exp	frac
0	**10001100**	**11011011011010000000000**

Denormalized Values

- Condition: **`exp`** = **`000…0`**

- Exponent value: ***E* = –*Bias* + 1** (instead of E = 0 – Bias)
- Significand coded with implied leading 0: **M = `0.xxx…x`**$_2$
    - **`xxx…x`**: bits of **`frac`**
- Cases
    - **`exp`** = 000…0, **`frac`** = 000…0
        - Represents value 0
        - Note distinct values: +0 and –0 (why?)
    - **`exp`** = 000…0, **`frac`** ≠ 000…0
        - Numbers very close to 0.0
        - Lose precision as get smaller
        - Equispaced

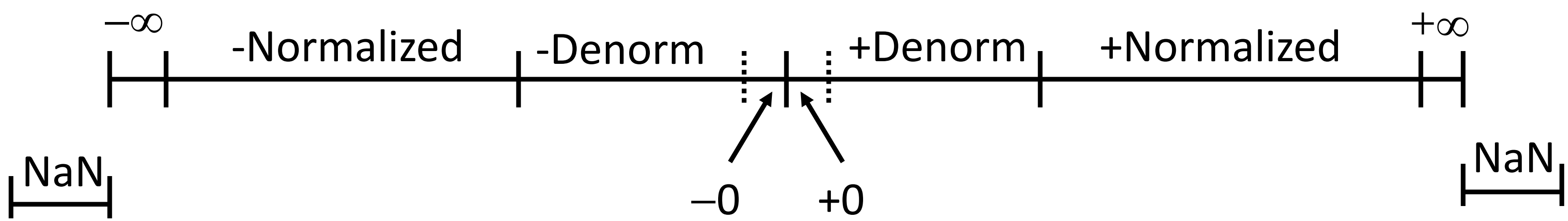
Special Values

- Condition: **`exp = 111…1`**

- Case: **`exp = 111…1, frac = 000…0`**
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g. 1.0/0.0 = -1.0/-0.0 = +∞, 1.0/-0.0 = -∞

- Case: **`exp`** = **`111…1, frac`** ≠ **`000…0`**
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), ∞ - ∞, ∞ * 0

Visualization: Floating Point Encodings



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Tiny Floating Point Example

s	exp	frac
1	4	3

- 8-bit floating point representation
  - the sign bit is in the most significant bit.
  - the next four bits are the exponent, with a bias of 7.
  - the last three bits are the **`frac`**

- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

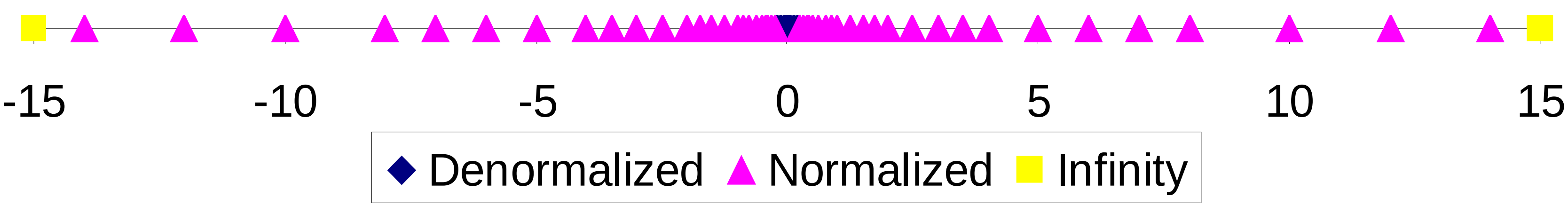
Dynamic range (positive only)

	s exp frac	E	Value	
	`s exp frac`	*E*	**Value**	
Denormalized numbers	`0 0000 000`	`-6`	`0`	
	`0 0000 001`	`-6`	`1/8*1/64 = 1/512`	closest to zero
	`0 0000 010`	`-6`	`2/8*1/64 = 2/512`	
	`…`			
	`0 0000 110`	`-6`	`6/8*1/64 = 6/512`	
	`0 0000 111`	`-6`	`7/8*1/64 = 7/512`	largest denorm
Normalized numbers	`0 0001 000`	`-6`	`8/8*1/64 = 8/512`	smallest norm
	`0 0001 001`	`-6`	`9/8*1/64 = 9/512`	
	`…`			
	`0 0110 110`	`-1`	`14/8*1/2 = 14/16`	
	`0 0110 111`	`-1`	`15/8*1/2 = 15/16`	closest to 1 below
	`0 0111 000`	`0`	`8/8*1 = 1`	
	`0 0111 001`	`0`	`9/8*1 = 9/8`	closest to 1 above
	`0 0111 010`	`0`	`10/8*1 = 10/8`	
	`…`			
	`0 1110 110`	`7`	`14/8*128 = 224`	
	`0 1110 111`	`7`	`15/8*128 = 240`	largest norm
	`0 1111 000`	`n/a`	`inf`	

Distribution of values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is $2^{3-1}-1 = 3$

s	exp	frac
1	3	2

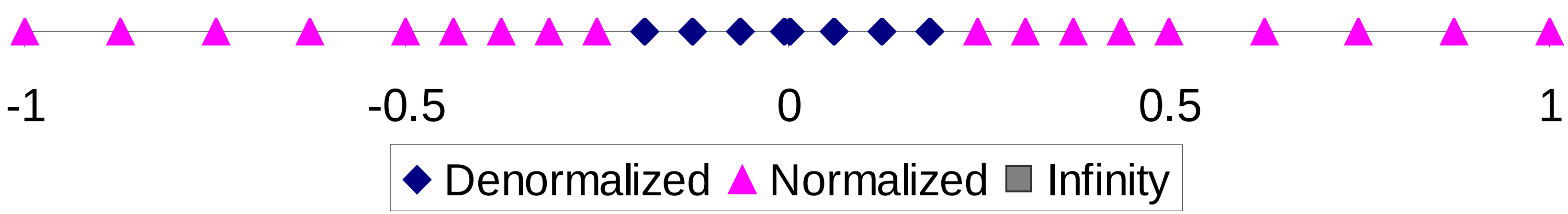


- Notice how the distribution gets denser toward zero.

Distribution of values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3

s	exp	frac
1	3	2



Interesting numbers

	Description	exp	frac	Numeric Value
•	Zero	00…00	00…00	0.0
•	Smallest pos. denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
	– Single ≈ 1.4×10^{-45}			
	– Double ≈ 4.9×10^{-324}			
•	Largest denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
	– Single ≈ 1.18×10^{-38}			
	– Double ≈ 2.2×10^{-308}			
•	Smallest pos. normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
	– Just larger than largest denormalized			
•	One	01…11	00…00	1.0
•	Largest normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
	– Single ≈ 3.4×10^{38}			
	– Double ≈ 1.8×10^{308}			

`{single,double}`

Special properties of encoding

- FP zero same as integer zero
  - All bits = 0 (for +0)

- Can (almost) use unsigned integer comparison
  - Must first compare sign bits
  - Must consider -0 = 0
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating point operations: basic idea

- $\mathbf{x} +_f \mathbf{y}$ = **Round(x + y)**
- $\mathbf{x} \times_f \mathbf{y}$ = **Round(x × y)**
- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into **`frac`**

Rounding

- Rounding modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	-$1.50
Towards zero	$1	$1	$1	$2	-$1
Round down (-∞)	$1	$1	$1	$2	-$2
Round up (+∞)	$2	$2	$2	$3	-$1
Nearest Even (default)	$1	$2	$2	$2	-$2

- What are the advantages of the modes?

Closer look at Round-To-Even

- Default rounding mode
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under- estimated
- Applying to other decimal places / bit positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

Value	Result	Description
1.2349999	1.23	(less than half way)
1.2350001	1.24	(greater than half way)
1.2350000	1.24	(half-way – round up)
1.2450000	1.24	(half way – round down)

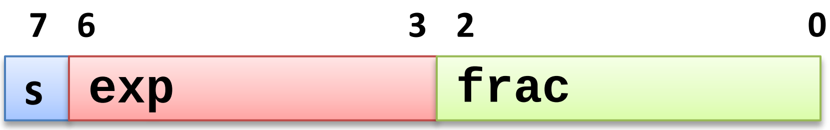
Rounding Binary Numbers

- Binary fractional numbers
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$
- Examples

  Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Result
$2\ {}^{3}/_{32}$	10.00011_2	10.00_2	< ½ : down	2
$2\ {}^{3}/_{16}$	10.00110_2	10.01_2	> ½ : up	$2\ {}^{1}/_{4}$
$2\ {}^{7}/_{8}$	10.11100_2	11.00_2	= ½ : up	3
$2\ {}^{5}/_{8}$	10.10100_2	10.10_2	= ½ : down	2 ½

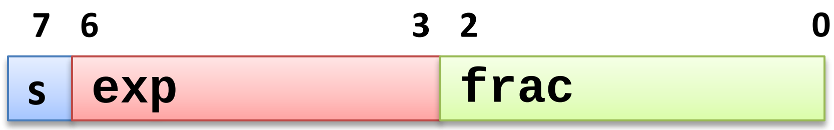
Creating a floating point number



- Steps
  - Normalize to have leading 1
  - Round to fit within fraction
  - Postnormalize to deal with effects of rounding
- Case study
  - Convert 8-bit unsigned numbers to tiny floating point format

Value	Binary
128	`10000000`
15	`00001101`
17	`00010001`
19	`00010011`
138	`10001010`
63	`00111111`

Normalize



Requirement

- Set binary point so that numbers of form 1.xxxxx
- Adjust all to have leading one
  - Decrement exponent as shift left

Value	Binary	Fraction	Exponent
128	10000000	1.0000000	7
15	00001101	1.1010000	3
17	00010001	1.0001000	5
19	00010011	1.0011000	5
138	10001010	1.0001010	7
63	00111111	1.1111100	5

Rounding

1.BBGRXXX

Guard bit: LSB of result

Round bit: 1st bit removed

Sticky bit: OR of remaining bits

- Round up conditions
  - Round = 1, Sticky = 1 ➔ > 0.5
  - Guard = 1, Round = 1, Sticky = 0 ➔ Round to even

Value	Fraction	GRS	Incr?	Rounded
128	1.0000000	000	N	1.000
15	1.1010000	100	N	1.101
17	1.0001000	010	N	1.000
19	1.0011000	110	Y	1.010
138	1.0001010	011	Y	1.001
63	1.1111100	111	Y	10.000

Postnormalize

- Issue
  - Rounding may have caused overflow
  - Handle by shifting right once & incrementing exponent

Value	Rounded	Exp	Adjusted	Result
128	1.000	7		128
15	1.101	3		15
17	1.000	4		16
19	1.010	4		20
138	1.001	7		134
63	10.000	5	1.000/6	64

FP Multiplication

$(-1)^{s1}\ M_1\ \ 2^{E1}\ \ \times\ \ (-1)^{s2}\ M_2\ \ 2^{E2}$

- Exact Result: $(-1)^s\ M\ \ 2^E$
  - Sign s: s_1 ^ s_2
  - Significand M: $M_1 * M_2$
  - Exponent E: $E_1 + E_2$

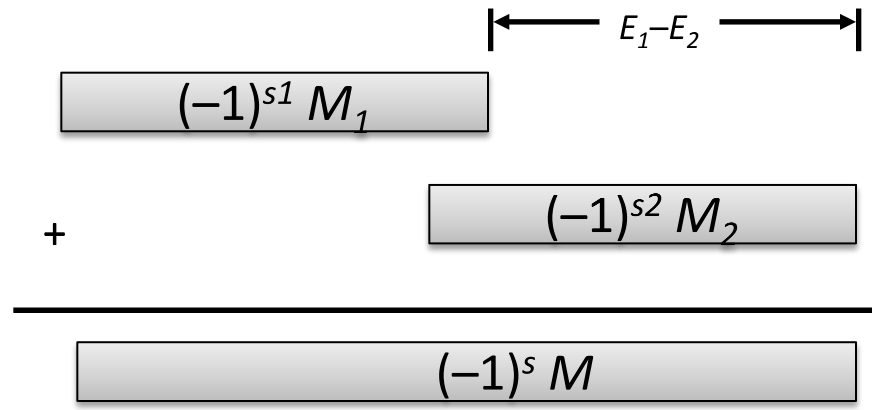
- Fixing
  - If $M \geq 2$, shift M right, increment E
  - If E out of range, overflow
  - Round M to fit **`frac`** precision

- Implementation
  - Biggest chore is multiplying significands

Floating Point Addition

$(-1)^{s1} M_1 \; 2^{E1} \;+\; (-1)^{s2} M_2 \; 2^{E2}$

Assume $E_1 > E_2$



- Exact Result: $(-1)^s M \; 2^E$
  - Sign s, significand M:
    - Result of signed align & add
  - Exponent E: E_1
- Fixing
  - If $M \geq 2$, shift M right, increment E
  - if $M < 1$, shift M left k positions, decrement E by k
  - Overflow if E out of range
  - Round M to fit **`frac`** precision

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Mathematical properties of floating point addition

- Compare to those of Abelian Group
  - Closed under addition? *Yes*
    - But may generate infinity or NaN
  - Commutative? *Yes*
  - Associative? *No*
    - Overflow and inexactness of rounding
  - 0 is additive identity? *Yes*
  - Every element has additive inverse? *Almost*
    - Except for infinities & NaNs

- Monotonicity
  - $a \geq b \Rightarrow a+c \geq b+c$? *Almost*
    - Except for infinities & NaNs

Mathematical properties of floating point multiplication

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- Compare to Commutative Ring
  - Closed under multiplication? *Yes*
    - But may generate infinity or NaN
  - Multiplication Commutative? *Yes*
  - Multiplication is Associative? *No*
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity? *Yes*
  - Multiplication distributes over addition? *No*
    - Possibility of overflow, inexactness of rounding

- Monotonicity
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? *Almost*
    - Except for infinities & NaNs

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Floating Point in C

- C Guarantees Two Levels

  `float` single precision

  `double` double precision

- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double/float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to **`TMin`**
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

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Summary

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- IEEE Floating Point has clear mathematical properties
- Represents numbers of form $M \times 2^E$
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers

Floating Point Puzzles

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- For each of these C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

- `x == (int)(float) x`
- `x == (int)(double) x`
- `f == (float)(double) f`
- `d == (float) d`
- `f == -(-f);`
- `2/3 == 2/3.0`
- `d < 0.0` ⇒ `((d*2) < 0.0)`
- `d > f` ⇒ `-f > -d`
- `d * d >= 0.0`
- `(d+f)-d == f`